On almost structural completeness

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Deductive systems

Sent - set of propositional sentences using connectives

Ax - axioms ($\subseteq Sent$)

inference rules: $\dfrac{\Delta}{\varphi}$, $\Delta \subseteq_{fin} Sent$, $\varphi \in Sent$

only structural (schematic) rules - closed on substitutions

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deductive system (Ax, R)

We identify deductive systems with finitary structural consequence relations $\vdash \subseteq \mathcal{P}(Sent) \times Sent$

$$(Ax, R) \mapsto \vdash_{Ax,R}$$

We write $\vdash_L$ for some known systems like S4, S5, i.e. $\vdash_{S4}$ $\vdash_{S5}$ etc.

Deductive systems, Structural Completeness

A rule $r: \quad \dfrac{\Delta}{\varphi},$ is

-admissible in $\vdash$, if

$$\vdash \sigma(\psi), \text{ for each } \psi \in \Delta, \text{ implies } \vdash \sigma(\varphi)$$

for each substitution σ;

-derivable in $\vdash$ if

$$\Delta \vdash \varphi$$

A deductive system is Structurally Complete (SC) if every rule admissible in $\vdash$ is derivable (Pogorzelski '71).

Theorem (Makinson '76)

A deductive system $\vdash$ is SC if it cannot be extended without extending theorems, i.e. $\vdash$ is maximal among all $\vdash'$ having the same set of theorems as $\vdash$.

Deductive systems, Structural Completeness

Examples SC

Classical logic,
Gödel -Dummett logic LC, Gödel logics G_n (D.-Wronski),
positive Łukasiewicz logics (Wojtylak),
$\mathrm{INT}^{\rightarrow}$, Medvedev logic (Prucnal),
S4.3Grz (Rybakov),
Product fuzzy logic, fragments of fuzzy logic (Metcalfe),
fragments of relevant logic (Raftery) ...

Examples Not SC

INT, S4, S5, all Łukasiewicz logics $\mathfrak{L}_n$, $\mathfrak{L}_\infty$ ...

Reasons?
INT, S4, $\mathfrak{L}_\infty$ - "serious";
S5, $\mathfrak{L}_n$ - "less serious". Why? The only reason: passive rules.

Passive rules, ASC

$r : \Delta/\beta$ is passive in $\vdash$ if

$$(\forall\tau)(\exists\delta \in \Delta)\ \nvdash \tau(\delta)$$

(Δ is not unifiable in $\vdash$)

Example: S5, S4.3

("less serious", paradox of "$\Rightarrow$" - admissible):

$$P_2 : \quad \frac{\Diamond x \wedge \Diamond\neg x}{\bot}; \qquad (\forall x)((\Diamond x \wedge \Diamond\neg x \approx \top) \to \bot \approx \top)$$

$\nvdash_{S5} \Diamond\tau(x) \wedge \Diamond\neg\tau(x)$ all τ, but $\Diamond x \wedge \Diamond\neg x$ is consistent in S5, S4.3

A deductive system $\vdash$ is almost structurally complete, ASC iff every admissible rule in $\vdash$ which is not passive is derivable in $\vdash$.

Projective Unification

A substitution σ is called a unifier for terms t_1, t_2 in an equational theory T if $\vdash_T \sigma(t_1) \approx \sigma(t_2)$.
Such terms t_1, t_2 are called unifiable in T.

IN LOGIC - Deductive Systems:

A substitution σ is a unifier for a formula φ in $\vdash$, if $\vdash \sigma(\varphi)$,

A formula φ is unifiable in $\vdash$ if it has a unifier.
A unifier σ for φ is called projective if

$$\varphi \vdash \sigma(x) \leftrightarrow x, \text{ for all } x \in var(\varphi)$$

(Ghilardi '99).

A deductive system enjoys projective unification if every unifiable formula has a projective unifier.

Almost Structural Completeness via Unification

Fact (D. '11)

If a deductive system has projective unification then it is ASC. In particular, every discriminator variety is ASC.

Examples ASC $\setminus$ SC

n-potent Basic Fuzzy Logics, Łukasiewicz logics $\mathfrak{L}_n$ (D. '06),
modal logics: S5, all NExt S4.3 (D.-Wojtylak '11),
relation algebras
more coming...

Fact

For every $\vdash$ there exists its SC extension $\vdash^{SC}$ with the same set of theorems.

How far (algebraically) is $\vdash^{SC}$ from $\vdash$?

- for Łukasiewicz logics $\mathfrak{L}_n$ take a product with $\mathbf{2}$, i.e., $\mathbf{L}_n \times \mathbf{2}$.

Quasivarieties

Quasi-identities look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{t}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \rightarrow s(\bar{x}) \approx t(\bar{x})$$

Quasivarieties look like

$$\mathrm{Mod(quasi\text{-}identities)}$$

Correspondence for algebraizable deductive systems

deductive system $\vdash$	$\leftrightsquigarrow$	quasivariety $\mathcal{Q}_{\vdash}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities
inference rules	$\leftrightsquigarrow$	quasi-identities

Almost Structural completeness algebraically

F - $\mathcal{Q}$-algebra over $\aleph_0$ generators
Q(**F**) - quasivariety generated by **F**

A quasivariety $\mathcal{Q}$ is SC if $\mathcal{Q} = \mathrm{Q}(\mathbf{F})$, i.e., every quasi-identity valid in **F** is valid in $\mathcal{Q}$ too.

$\mathcal{Q}$ is ASC if for every quasi-identity q valid in **F** either q is valid in $\mathcal{Q}$ or its premises are not satisfiable in **F**,
i.e., every non-passive quasi-identity valid in **F** is also valid in $\mathcal{Q}$.

Theorem

The following conditions are equivalent:

- $\mathcal{Q}$ is ASC;
- For every $\mathbf{A} \in \mathcal{Q}$, $\mathbf{A} \times \mathbf{F} \in \mathrm{Q}(\mathbf{F})$ (Metcalfe, Röthlisberger '13)
- For every $\mathbf{A} \in \mathcal{Q}_{SI}$, $\mathbf{A} \times \mathbf{F} \in \mathrm{Q}(\mathbf{F})$
- For every $\mathbf{A} \in \mathcal{Q}$, $(\exists h\colon \mathbf{A} \to \mathbf{F})$ yields $\mathbf{A} \in \mathrm{Q}(\mathbf{F})$
- For every $\mathbf{A} \in \mathcal{Q}_{FP}$, $(\exists h\colon \mathbf{A} \to \mathbf{F})$ yields $\mathbf{A} \in \mathrm{Q}(\mathbf{F})$

Consequences

Corollary

Every variety with projective unification is ASC. This includes discriminator varieties ($\mathcal{S}5$, $\mathcal{MV}_n$) and a bit more (e.g. $\mathcal{S}4.3$) (already mentioned by Wojtek).

Corollary (Metcalfe, Röthlisberger '13)

There is an efficient algorithm for deciding whether a finitely generated quasivariety is ASC.

Corollary

If every nontrivial algebra from $\mathcal{Q}$ admits a homomorphism into $\mathbf{F}$, then $\mathcal{Q}$ is ASC iff it is SC.
Examples: idempotent elements, Heyting algebras, Grzegorczyk algebras.

Better characterization for ASC

Theorem

Assume that $\mathcal{Q}$ is a quasivariety with finite model property and equationally definable relative principal congruences. Assume that $\mathbf{F}$ has a simple finite subalgebra $\mathbf{C}$. Then $\mathcal{V}$ is ASC iff for every $\mathbf{S} \in \mathcal{V}_{SI}$

$$\mathbf{S} \leqslant \mathbf{F} \quad \text{or} \quad \mathbf{S} \times \mathbf{C} \leqslant \mathbf{F}.$$

- Equational definability of relative principal congruences corresponds to deduction-detachment theorem.
- in many cases $\mathbf{C}$ is is a 2-element Boolean algebra with extra operations.
- In order to use this theorem we need to know the structure of free and SI algebras in $\mathcal{V}$.

Discriminator varieties revisited

Example

Let $\mathbf{L}_n$ be the $(n+1)$-element chain MV-algebra and $\mathcal{MV}_n = \mathrm{V}(\mathbf{L}_n)$. Since

$$\mathbf{F}(m) \cong \prod_{k|n} {\mathbf{L}_k}^{c_k}$$

for some $c_k > 0$, and $\mathbf{L}_1 \leqslant \mathbf{L}_k$,
we have $\mathbf{L}_k \times \mathbf{L}_1 \leqslant \mathbf{F}$
(proved already in '82 by Pogorzelski and Wojtylak in logic)
and $\mathcal{MV}_n$ is ASC.

Discriminator varieties revisited II

Example

Let $\mathbf{A}_n$ be the monadic algebras with n-atoms and only 0 and 1 closed. Let $\mathcal{S}5 = \mathsf{V}(\mathbf{A}_n \mid n > 0)$ be the variety of monadic algebras. Since

$$\mathbf{F}(m) \cong \prod_{k=1}^{2^m} \mathbf{A}_k{}^{c_k}$$

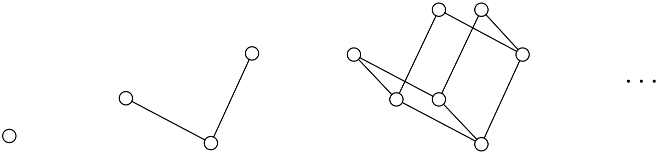
for some $c_k > 0$, and $\mathbf{A}_1 \leqslant \mathbf{A}_k$, we have $\mathbf{A}_k \times \mathbf{A}_1 \leqslant \mathbf{F}$ and $\mathcal{S}5$ is ASC.

Example

Actually the argument for $\mathcal{MV}_n$ and $\mathcal{S}5$ works in a more general situation: for every locally finite discriminator variety $\mathcal{V}$ with a nontrivial algebra embeddable into nontrivial members of $\mathcal{V}$.

New ASC \ SC varieties of modal algebras

In known examples of ASC \ SC varieties of modal algebras finitely presented algebras admitting a homomorphism into **F** embed into **F** - they have projective unification.



$\mathbf{Lev}_1$ $\mathbf{Lev}_2$ $\mathbf{Lev}_3$ $\cdots$

$$\mathcal{LEV}_n = \mathsf{V}(\mathbf{Lev}_n^+) \qquad \mathcal{MED} = \bigvee \mathcal{LEV}_n$$

Theorem

Let $\mathcal{V} \in \{\mathcal{LEV}_2, \mathcal{LEV}_3, \ldots, \mathcal{MED}\}$ and $\mathcal{W}$ be a non-minimal subvariety of $\mathcal{S}5$. Then the varietal join $\mathcal{V} \vee \mathcal{W}$ is ASC \ SC.

New ASC \ SC varieties of modal algebras II

Proof

- $\mathcal{V}$ is SC (Prucnal '76),
- $\mathcal{W}$ is ASC $\setminus$ SC,
- SI algebras from $\mathcal{V} \vee \mathcal{W}$ are either in $\mathcal{V}$ or in $\mathcal{W}$,
- $\mathbf{F}(n) \cong \mathbf{F}_V(n) \times \mathbf{G}_{\mathcal{W}}(n)$, where $\mathbf{G}_{\mathcal{W}}(n)$ is a factor of $\mathbf{F}_{\mathcal{W}}(n)$,
- 2-element modal algebra embeds into every nontrivial modal algebra.

Theorem

Let $\mathcal{V} \in \{\mathcal{LEV}_2, \mathcal{LEV}_3, \ldots, \mathcal{MED}\}$ and $\mathcal{W}$ be a non-minimal subvariety of $\mathcal{S}5$. Then the varietal join $\mathcal{V} \vee \mathcal{W}$ has a finitely presented algebra admitting a homomorphism into $\mathbf{F}$ and is not embeddable into $\mathbf{F}$. Hence it has neither unitary nor projective unification.

The end

This is all

Thank you!